

Directed random growth models on the plane

Timo Seppäläinen *

February 1, 2008

Abstract

This is a brief survey of laws of large numbers, fluctuation results and large deviation principles for asymmetric interacting particle systems that represent moving interfaces on the plane. We discuss the exclusion process, the Hammersley process and the related last-passage growth models.

1 Introduction

This article is a brief overview of recent results for a class of stochastic processes that represent growth or motion of an interface in two-dimensional Euclidean space. The models discussed have in a sense rather orderly evolutions, and the word “directed” is included in the title to evoke this feature.

Let us begin with generalities about these stochastic processes. The state at time $t \in [0, \infty)$ is of the form $h(t) = (h_i(t) : i \in \mathbb{Z})$ with the interpretation that the integer- or real-valued random variable $h_i(t)$ represents the height of the interface over site i of the substrate \mathbb{Z} . We call the state $h = (h_i)$ a *height function* on \mathbb{Z} . The interface on the plane is then represented by the graph $\{(i, h_i) : i \in \mathbb{Z}\}$. Each particular process has a state space that defines the set of admissible height functions. The state space will be defined by putting restrictions on the increments (discrete derivatives) $h_i - h_{i-1}$ of the height functions.

The random dynamics of the state are specified by the *jump rates* of the individual height variables h_i . The rates are functions of the current state h . For the sake of illustration, suppose that only jumps ± 1 are permitted for each variable h_i . Then the model is defined by giving two functions $p(h)$ and $q(h)$. If the current state is h , then the height value h_0 at the origin jumps down to a new value $h'_0 = h_0 - 1$ with rate $p(h)$, and jumps up to a new value $h'_0 = h_0 + 1$ with rate $q(h)$. In the spatially homogeneous case the rates for h_i are $p(\theta_i h)$ and $q(\theta_i h)$ where the spatial translations are defined by $(\theta_i h)_j = h_{i+j}$.

The quantity $p(h)$ is the rate for the jump $h_0 \leadsto h_0 - 1$ in an instantaneous sense: in the current state h , the probability that this jump happens in the next infinitesimal time interval $(0, dt)$ is $p(h)dt + o(dt)$. Rigorous constructions of the processes utilize Poisson processes or “Poisson clocks.” A rate λ Poisson

*Research supported in part by NSF Grant DMS-0701091.

process $N(t)$ is a simple continuous time Markov chain: it starts at $N(0) = 0$, runs through the integers $0, 1, 2, 3, \dots$ in increasing order, and waits for a rate λ exponential random time between jumps. A rate λ exponential random time is defined by its density $\varphi(t) = \lambda e^{-\lambda t}$ on \mathbb{R}_+ . The number of jumps $N(s+t) - N(s)$ in time interval $(s, s+t]$ has the mean λt Poisson distribution

$$P\{N(s+t) - N(s) = k\} = \frac{e^{-\lambda t} (\lambda t)^k}{k!} \quad (k \geq 0).$$

If the overall rates are bounded, say by $p(h) \leq \lambda$, then a Poisson clock with a time-varying rate $p(h(t))$ can be obtained from $N(t)$ by randomly accepting a jump at time t with probability $p(h(t))/\lambda$.

Later we mention in passing rigorous constructions of some processes. In each case the outcome of the construction is that all the random variables $\{h_i(t) : i \in \mathbb{Z}, t \geq 0\}$ are defined as measurable functions on an underlying probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Since these processes evolve through jumps, the appropriate path regularity is that with probability 1, the path $t \mapsto h(t)$ is right-continuous with left limits (cadlag for short). The use of Poisson clocks makes the stochastic process $h(t)$ a *Markov process*. This means that if the present state $h(t)$ is known, the future evolution $(h(s) : s > t)$ is statistically independent of the past $(h(s) : 0 \leq s < t)$. This is a consequence of the “forgetfulness property” of the exponential distribution. For a complete discussion of these foundational matters we must refer to textbooks on probability theory and stochastic processes.

This article covers only asymmetric systems. Asymmetry in this context means that the height variables $h_i(t)$ on average tend to move more in one direction than the other. For definiteness, we define the models so that the downward direction is the preferred one. In fact, the great majority of the paper is concerned with *totally asymmetric systems* for which $q(h) \equiv 0$, in other words only downward jumps are permitted. Symmetric systems behave quite differently from asymmetric systems, hence restricting treatment to one or the other is natural.

Stochastic processes with a large number of interacting components such as the height process $h(t) = (h_i(t) : i \in \mathbb{Z})$ belong in an area of probability theory called *interacting particle systems*. (Spitzer, 1970) is one of the seminal papers of this subject. Here is a selection of books and lecture notes on the topic: De Masi and Presutti (1991), Durrett (1988), Kipnis and Landim (1999), Liggett (1985), Liggett (1999), Liggett (2004), Varadhan (2000). Krug and Spohn (1992) and Spohn (1991) are sources that combine mathematics and the theoretical physics side.

Our treatment is organized around three basic questions posed about stochastic models: (i) laws of large numbers, (ii) fluctuations, and (iii) large deviations.

(i) *Laws of large numbers* give deterministic limit shapes and evolutions under appropriate space and time scaling. A parameter $n \nearrow \infty$ gives the ratio of macroscopic and microscopic scales. A sequence of processes $h^n(t)$ indexed

by n is considered. Under appropriate hypotheses the height process satisfies this type of result: for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$

$$n^{-1}h_{[nx]}^n(nt) \rightarrow u(x, t) \quad \text{as } n \rightarrow \infty, \quad (1)$$

and the limit function u satisfies a Hamilton-Jacobi equation $u_t + f(u_x) = 0$.

(ii) *Fluctuations.* After a law of large numbers the next question concerns the random fluctuations around the large scale behavior. One seeks an exponent α that describes the magnitude of these fluctuations, and a precise description of them in the limit. A typical statement would be

$$\frac{h_{[nx]}^n(nt) - nu(x, t)}{n^\alpha} \longrightarrow Z(t, x) \quad (2)$$

where $Z(t, x)$ is a random variable whose distribution would be described as part of the result. The convergence is of a weak type, where it is the probability distribution of the random variable on the left that converges.

(iii) *Large deviations.* The vanishing probabilities of atypical behavior fall under this rubric. Often these probabilities decay as e^{-Cn^β} to leading order, with another exponent $\beta > 0$. The precise constant $C \in (0, \infty)$ is also of interest and comes in the form of a *rate function*. When all the ingredients are in place the result is called a large deviation principle (LDP). An LDP from the law of large numbers (1) with rate function $I : \mathbb{R} \rightarrow [0, \infty]$ could take this form:

$$\lim_{\varepsilon \searrow 0} \lim_{n \rightarrow \infty} n^{-\beta} \log \mathbf{P}\{h_{[nx]}^n(nt) \in (z - \varepsilon, z + \varepsilon)\} = -I(z) \quad (3)$$

valid for points z in some range. Positive values $I(z) > 0$ represent atypical behavior, while limit (1) would force $I(u(x, t)) = 0$.

Example. For classical examples of these statements let us consider one-dimensional nearest-neighbor random walk. Fix a parameter $0 < p < 1$. Let $\{X_k\}$ be independent, identically distributed (IID) ± 1 -valued random variables with common distribution $\mathbf{P}\{X_k = 1\} = p = 1 - \mathbf{P}\{X_k = -1\}$. Define the random walk by $S_0 = 0$, $S_n = S_{n-1} + X_n$ for $n \geq 1$. Then the *strong law of large numbers* gives the long term velocity:

$$\lim_{n \rightarrow \infty} n^{-1}S_n = v \quad \text{where } v = \mathbf{E}(X_1) = 2p - 1.$$

The convergence in the limit above is almost sure (a.s.), that is, almost everywhere (a.e.) on the underlying probability space of the variables $\{X_k\}$.

The order of nontrivial fluctuations around the limit is $n^{1/2}$ (“diffusive”) and in the limit these fluctuations are Gaussian. That is the content of the *central limit theorem*:

$$\lim_{n \rightarrow \infty} \mathbf{P}\left\{\frac{S_n - nv}{\sigma n^{1/2}} \leq s\right\} = \Phi(s) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^s e^{-z^2/2} dz.$$

The parameter $\sigma^2 = \mathbf{E}[(X_1 - v)^2]$ is the variance.

Random walk satisfies this LDP:

$$\lim_{\varepsilon \searrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbf{P}\{|S_n - nx| \leq n\varepsilon\} = -I(x) \quad (4)$$

where the rate function $I: \mathbb{R} \rightarrow [0, \infty]$ is identically ∞ outside $[-1, 1]$ and

$$I(x) = \frac{1-x}{2} \log \frac{1-x}{2(1-p)} + \frac{1+x}{2} \log \frac{1+x}{2p} \quad \text{for } x \in [-1, 1]. \quad (5)$$

$I(x)$ can be interpreted as an entropy. Convex analysis plays a major role in large deviation theory. Part of the general theory behind this simple case is that I is the convex dual of the logarithmic moment generating function $\Lambda(\theta) = \log \mathbf{E}(e^{\theta X_1})$.

Results for random walk are covered in graduate probability texts such as (Durrett, 2004) and (Kallenberg, 2002).

At the outset we delineated the class of models discussed. Important models left out include *diffusion limited aggregation* (DLA) and *first-passage percolation*. Their interfaces are considerably more complicated than interfaces described by height functions. But even for the models discussed our treatment is not a complete representation of the mathematical progress of the past decade. In particular, this article does not delve into the recent work on Tracy-Widom fluctuations, Airy processes and determinantal point processes. These topics are covered by many authors, and we give a number of references to the literature in Sections 3.1 and 3.2. Overall, the best hope for this article is that it might inspire the reader to look further into the references.

Recurrent notation. The set of nonnegative integers is $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$, while $\mathbb{N} = \{1, 2, 3, \dots\}$. The integer part of a real x is $[x] = \max\{n \in \mathbb{Z} : n \leq x\}$. $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$.

Acknowledgement. The author thanks M. Balázs for valuable consultation during the preparation of this article.

2 Limit shape and evolution

We begin with the much studied *corner growth model* and a description that is not directly in terms of height variables. Attach nonnegative weights $\{Y_{i,j}\}$ to the points (i, j) of the positive quadrant \mathbb{N}^2 of \mathbb{Z}^2 , as in Figure 1. $Y_{i,j}$ represents the time it takes to occupy point (i, j) after the points to its left and below have been occupied. Assume that everything outside the positive quadrant is occupied at the outset so the process can start. Once occupied, a point remains occupied. Thus this is a totally asymmetric growth model, for the growing cluster never loses points, only adds them.

Let $G(k, \ell)$ denote the time when point (k, ℓ) becomes occupied. The above explanation is summarized by these rules: $G(k, \ell) = 0$ for $(k, \ell) \notin \mathbb{N}^2$, and

$$G(k, \ell) = G(k-1, \ell) \vee G(k, \ell-1) + Y_{k,\ell} \quad \text{for } (k, \ell) \in \mathbb{N}^2. \quad (6)$$

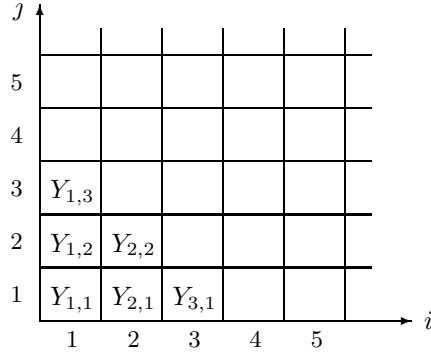


Figure 1: Each point $(i, j) \in \mathbb{N}^2$ has a weight $Y_{i,j}$ attached to it.

The last equality can be iterated until the corner $(1, 1)$ is reached, resulting in this last-passage formula for G :

$$G(k, \ell) = \max_{\pi \in \Pi_{k, \ell}} \sum_{(i, j) \in \pi} Y_{i, j} \quad (7)$$

where $\Pi_{k, \ell}$ is the collection of nearest-neighbor up-right paths π from $(1, 1)$ to (k, ℓ) . Figure 2 represents one such path for $(k, \ell) = (5, 4)$.

This model and others of its kind are called *directed last-passage percolation models*. “Directed” refers to the restrictions on admissible paths, and “last-passage” to the feature that the occupation time $G(k, \ell)$ is determined by the slowest path to (k, ℓ) . (By contrast, in first-passage percolation occupation times are determined by quickest paths.)

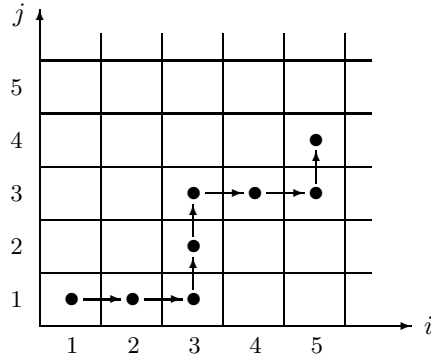


Figure 2: An admissible path from $(1, 1)$ to $(5, 4)$

Our first goal is to argue the existence of a limit for $n^{-1}G([nx], [ny])$ as

$n \rightarrow \infty$. Assume now that the weights $\{Y_{i,j}\}$ are IID nonnegative random variables.

The idea is to exploit sub(super)additivity. Generalize the definition of $G(k, \ell)$ to

$$G((k, \ell), (m, n)) = \max_{\pi \in \Pi_{(k, \ell), (m, n)}} \sum_{(i, j) \in \pi} Y_{i, j},$$

where $\Pi_{(k, \ell), (m, n)}$ is the collection of nearest-neighbor up-right paths π from $(k+1, \ell+1)$ to (m, n) . The definitions lead to the superadditivity

$$G(k, \ell) + G((k, \ell), (m, n)) \leq G(m, n). \quad (8)$$

Kingman's subadditive ergodic theorem (Durrett, 2004, Chapter 6) and some estimation implies the existence of a deterministic limit

$$\gamma(x, y) = \lim_{n \rightarrow \infty} n^{-1} G([nx], [ny]) \quad \text{for } (x, y) \in \mathbb{R}_+^2. \quad (9)$$

Moment assumptions under which the limit function γ is continuous up to the boundary were investigated by Martin (2004). We turn to the problem of computing γ explicitly, and for this we need very specialized assumptions. Essentially only one distribution can be currently handled: the exponential, and its discrete counterpart, the geometric. Take the $\{Y_{i,j}\}$ to be IID rate 1 exponential random variables. In other words their common density is e^{-y} .

The difficulty with finding the explicit limit has to do with the superadditivity. The limit in Birkhoff's ergodic theorem is simply the expectation of the function averaged over shifts: $n^{-1} \sum_{k=1}^n f \circ \theta_k \rightarrow \mathbf{E}f$ (Durrett, 2004, Chapter 6). But the subadditive ergodic theorem gives only an asymptotic expression for the limit. We need a new ingredient. We shall embed the last-passage model into the totally asymmetric simple exclusion process (TASEP). This has explicitly identifiable invariant distributions ("steady states") with which we can do explicit calculations.

Originally TASEP was introduced as a particle model. We wish to link TASEP with the last-passage model in a way that preserves the original formulation of TASEP, while mapping particle occupation variables into height increments and particle current into column growth. To achieve this we transform the coordinates (i, j) of Figure 1 via the bijection $(i, j) \mapsto (i-j, -j)$. The result is the last-passage model of Figure 3. Weights are relabeled as $X_{i,j} = Y_{i-j, -j}$. The transformation of admissible paths is illustrated by Figure 4. Let the new last-passage times be denoted by $H(k, \ell)$. For $\ell < 0 \wedge k$ the maximizing-path formulation uses now paths of the kind represented in Figure 4:

$$H(k, \ell) = \max_{\sigma \in \Sigma_{k, \ell}} \sum_{(i, j) \in \sigma} X_{i, j} \quad (10)$$

where $\Sigma_{k, \ell}$ is the collection of paths σ from $(0, -1)$ to (k, ℓ) that take steps of two types: $(1, 0)$ and $(-1, -1)$. The connection with the previous last-passage

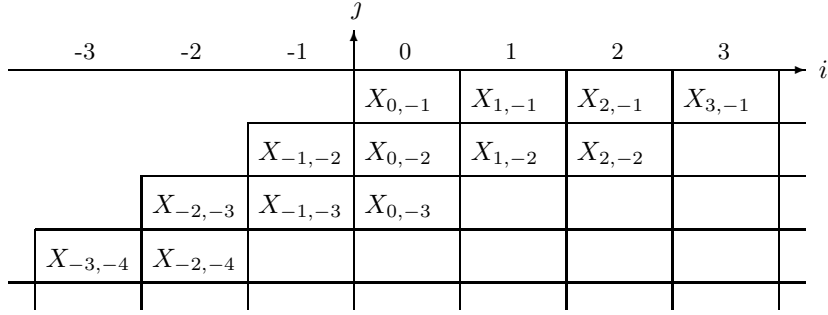


Figure 3: Last-passage model for TASEP. The horizontal i -axis and the vertical j -axis are labeled, and points on the i -axis from -3 to 3 are marked. Weights $X_{i,j}$ are attached to points (i, j) such that $i \in \mathbb{Z}$, $j \in -\mathbb{N}$ and $j < 0 \wedge i$.

process is $H(k, \ell) = G(k - \ell, -\ell)$. The process $\{H(k, \ell)\}$ is also defined by the recursion

$$H(k, \ell) = H(k - 1, \ell) \vee H(k + 1, \ell + 1) + X_{k, \ell} \quad (\ell < 0 \wedge k) \quad (11)$$

together with the boundary values $H(k, \ell) = 0$ for $\ell \geq 0 \wedge k$. It satisfies the limit

$$\lambda(x, y) = \lim_{n \rightarrow \infty} n^{-1} H([nx], [ny]) = \gamma(x - y, -y) \quad \text{for } y < 0 \wedge x. \quad (12)$$

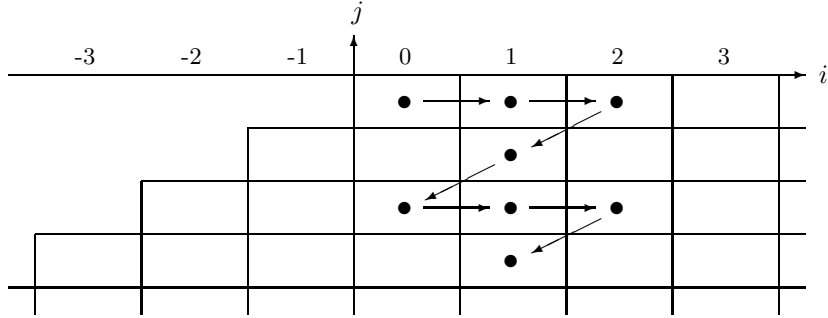


Figure 4: The image of the path in Figure 2. Now it goes from $(0, -1)$ to $(1, -4)$.

To establish the TASEP connection we first define a height process $w(t) = (w_i(t) : i \in \mathbb{Z})$ that will turn out to be an alternative description of the last-passage process $\{H(i, j)\}$. Initially at time $t = 0$ the height is given by

$$w_i(0) = \begin{cases} i, & i \leq -1 \\ 0, & i \geq 0. \end{cases} \quad (13)$$

This is the boundary of the region $\{j < 0 \wedge i\}$ filled with $X_{i,j}$'s in Fig. 3. This initial shape is a wedge, hence the symbol w .

Give each column i an independent rate 1 Poisson clock N_i . Variable w_i jumps downward according to this rule: if t is a jump time for Poisson process N_i , then

$$w_i(t) = w_i(t-) - 1 \text{ provided } \begin{cases} w_{i-1}(t-) = w_i(t-) - 1 \text{ and} \\ w_{i+1}(t-) = w_i(t-). \end{cases} \quad (14)$$

Equivalently, each column variable w_i jumps down independently at rate 1, as long as the state $w(t)$ remains in the state space

$$\mathcal{X}_1 = \{h \in \mathbb{Z}^{\mathbb{Z}} : h_i - h_{i-1} \in \{0, 1\} \text{ for all } i \in \mathbb{Z}\}. \quad (15)$$

(See Fig. 5 for an example.) The *interaction* between the variables is encoded in rule (14). It forces the time-evolution of each variable w_i to depend on the evolution of its neighbors. By contrast, without the interaction the variables w_i would simply march along as Poisson processes independently of each other.

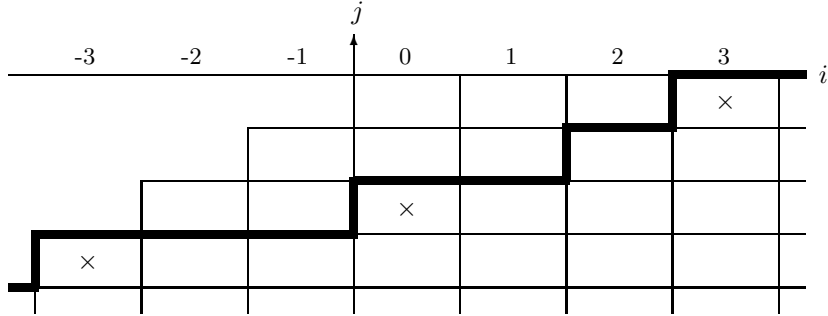


Figure 5: A possible height function $w(t)$ (thickset graph) with column values $w_{-1}(t) = -3$, $w_0(t) = -2$, $w_1(t) = -2$, etc. 8 jumps have taken place during time $(0, t]$. The columns grow downward. \times 's mark the allowable jumps from this state.

This construction defines the height process $w(t) = (w_i(t) : i \in \mathbb{Z})$ for all times $t \in [0, \infty)$ in terms of the family of Poisson clocks $\{N_i\}$. Given this process $w(t)$ define the stopping times

$$T(i, j) = \inf\{t \geq 0 : w_i(t) \leq j\} \quad (16)$$

that mark the time when column i first reaches level j . *Stopping time* is a technical term for a random time whose arrival can be verified without looking into the future.

Initial condition (13) implies $T(i, j) = 0$ for $j \geq i \wedge 0$. Rule (14) tells us that $T(i-1, j) \vee T(i+1, j+1)$ is the stopping time at which the system is ready

for w_i to jump from level $j + 1$ to j . (Note that w_i must have reached level $j + 1$ already earlier because if the rules are followed, $T(i + 1, j + 1)$ comes after $T(i, j + 1)$.) By the forgetfulness property of the exponential distribution, after the stopping time $T(i - 1, j) \vee T(i + 1, j + 1)$ it takes another independent rate 1 exponential time $\tilde{X}_{i,j}$ until w_i jumps from $j + 1$ to j . Consequently the process $\{T(i, j)\}$ satisfies the recursion

$$T(i, j) = T(i - 1, j) \vee T(i + 1, j + 1) + \tilde{X}_{i,j}. \quad (17)$$

This is of the same form as the recursion (11) satisfied by $\{H(i, j)\}$. From this one can prove that indeed the processes $\{T(i, j)\}$ and $\{H(i, j)\}$ are equal in distribution. Therefore (12) gives also $n^{-1}T([nx], [ny]) \rightarrow \lambda(x, y)$. This is the precise meaning of the earlier claim that the height process $w(t)$ gives an alternative description of the last-passage process $\{H(i, j)\}$.

Subadditivity and some estimation justifies the existence of a concave function g on \mathbb{R} such that

$$\lim_{t \rightarrow \infty} t^{-1}w_{[xt]}(t) = g(x) \quad \text{a.s. for } x \in \mathbb{R}. \quad (18)$$

Since rates are 1, g records only the initial height outside the interval $[-1, 1]$, and so

$$g(x) = 0 \wedge x \text{ for } |x| > 1.$$

Since the interface is a level curve of passage times, $\lambda(x, g(x)) = 1$ for $-1 \leq x \leq 1$. The last-passage limits are homogeneous in the sense that $\lambda(cx, cy) = c\lambda(x, y)$ for $c > 0$. Consequently λ and then γ can be obtained from g .

To summarize, thus far we have converted the original task of computing $\gamma(x, y)$ of (9) to finding the function g of (18) on the interval $[-1, 1]$. Now consider the general height process $h(t)$ with state space \mathcal{X}_1 from (15) and dynamics defined as for $w(t)$ above: height variable h_i jumps one step down at every jump epoch of the Poisson clock N_i , provided this jump does not take the height function out of \mathcal{X}_1 . A jump attempt that would violate the state space restriction is simply ignored.

Certain technical issues may trouble the reader. An infinite family of Poisson clocks has infinitely many jumps in any nonempty time interval $(0, \varepsilon)$. So there is no first jump attempt in the system and it is not obvious that the local rule leads to a well-defined global evolution: to determine the evolution of h_i on $[0, t]$ we need to look at the evolution of its neighbors $h_{i \pm 1}$, and then their neighbors, ad infinitum. However, given any $T < \infty$, almost surely there are indices $i_k \searrow -\infty$ and $i'_k \nearrow \infty$ such that N_{i_k} and $N_{i'_k}$ have no jumps during $(0, T]$. Consequently the system decomposes into (random) finite pieces that do not communicate before time T . The evolution can be determined separately in each finite segment which do experience only finitely many jumps up to time T (again almost surely). Another technical point is that the clocks $\{N_i\}$ have no simultaneous jumps (almost surely) so one never needs to consider more than one jump at a time.

Given that the height process $h(t)$ has been constructed, next define the increment process $\eta(t) = (\eta_i(t) : i \in \mathbb{Z})$ by

$$\eta_i(t) = h_i(t) - h_{i-1}(t). \quad (19)$$

Process $\eta(t)$ has compact state space $\{0, 1\}^{\mathbb{Z}}$ and its dynamics inherited from $h(t)$ can be succinctly stated as follows: each 10 pair becomes a 01 pair at rate 1, independently of the rest of the system. To see this connection, observe that if Poisson clock N_i jumps at time t , the height process undergoes the transformation $h_i(t) = h_i(t-) - 1$ only if $(\eta_i(t-) = 1, \eta_{i+1}(t-) = 0)$, and then after the jump the situation is $(\eta_i(t) = 0, \eta_{i+1}(t) = 1)$. This is a direct translation of the condition that jumps are executed only if the state h remains in the state space \mathcal{X}_1 .

It is natural to interpret the 1's as particles and the 0's as holes, or vacant sites. The process $\eta(t)$ is the *totally asymmetric simple exclusion process* (TASEP). In this model the only interaction between the particles is the exclusion rule that stipulates that particles are not allowed to jump onto occupied sites. This property is enforced by the evolution because the definitions made above ensure that a jump in Poisson clock N_i sends a particle from site i to site $i + 1$ only if site $i + 1$ is vacant. Total asymmetry refers to the property that particles jump only to the right, never left. The definitions also entail this connection between the heights and the particles:

$$h_i(0) - h_i(t) = \text{cumulative particle current across the edge } (i, i + 1). \quad (20)$$

We need to discuss two more properties of these processes, (i) stationary behavior and (ii) the envelope property. Then we are ready to compute the function g of (18).

(i) *Stationary behavior.* For $\rho \in [0, 1]$, the *Bernoulli* probability measure ν_ρ on $\{0, 1\}^{\mathbb{Z}}$ is defined by the requirement that

$$\nu_\rho\{\eta : \eta_i = 1 \text{ for } i \in I, \eta_j = 0 \text{ for } j \in J\} = \rho^{|I|}(1 - \rho)^{|J|} \quad (21)$$

for any disjoint $I, J \subseteq \mathbb{Z}$ with cardinalities $|I|$ and $|J|$. Measure ν_ρ corresponds to putting a particle at each site independently with probability ρ .

It is known that the measures $\{\nu_\rho\}_{\rho \in [0, 1]}$ are invariant for the process $\eta(t)$, and in fact they are the extremal members of the compact, convex set of invariant probability measures that are also invariant under spatial shifts. Invariance means that if the process $\eta(t)$ is started with a random ν_ρ -distributed initial state $\eta(0)$, then at each time $t \geq 0$ the state $\eta(t)$ is ν_ρ -distributed, and furthermore, the probability distribution of the entire process $\eta(\cdot) = (\eta(t) : t \geq 0)$ is invariant under time shifts.

If we know the current state $h(t)$, then the probability that h_0 jumps down in a short time interval $(t, t + \varepsilon)$ is $\varepsilon \eta_0(t)(1 - \eta_1(t)) + O(\varepsilon^2)$. This follows because a jump can happen only when a 10 pair is present, and from properties of Poisson processes. Estimation of this kind proves that

$$h_0(t) - h_0(0) = - \int_0^t \eta_0(s)(1 - \eta_1(s)) ds + M(t) \quad (22)$$

where $M(t)$ is a mean-zero martingale. This identity is a stochastic “fundamental theorem of calculus” of sorts. Since things are random the difference between $h_0(t) - h_0(0)$ and the integral of the infinitesimal rate cannot be identically zero. Instead it is a *martingale*. This is a process whose increments have mean zero in a very strong sense, namely even when conditioned on the entire past.

Let us average over (22) in the stationary situation. Let \mathbf{E}_{ν_ρ} denote expectation of functions of the stationary process $\eta(\cdot)$ whose state $\eta(t)$ is ν_ρ -distributed at each time t . Normalize the height process $h(\cdot)$ at time zero so that $h_0(0) = 0$. Then $h(\cdot)$ is entirely determined by $\eta(\cdot)$. Since $\eta_0(s)$ and $\eta_1(s)$ are independent at any fixed time s , we get

$$E_{\nu_\rho}[h_0(t)] = -tf(\rho) \quad (23)$$

where the *particle flux* is defined by

$$f(\rho) = \rho(1 - \rho). \quad (24)$$

(ii) *Envelope property*. Even though the flux f is nonlinear and therefore, as we see later, TASEP is governed by a nonlinear PDE, the height process has a valuable additivity property. Suppose a given initial height function $h(0) \in \mathcal{X}_1$ is the envelope of a countable collection $\{z^{(k)}(0)\}_{k \in \mathcal{K}}$ of height functions in the sense that

$$h_i(0) = \sup_{k \in \mathcal{K}} z_i^{(k)}(0) \quad \text{for each site } i \in \mathbb{Z}. \quad (25)$$

Take a single collection $\{N_i\}$ of Poisson clocks, and let all processes $h(t)$, $z^{(k)}(t)$ evolve from their initial height functions by following the same clocks $\{N_i\}$. This kind of simultaneous construction of many random objects for the purpose of comparison is called a *coupling*. By induction on jumps one can prove that this coupling preserves the envelope property for all time:

Lemma 2.1 $h_i(t) = \sup_{k \in \mathcal{K}} z_i^{(k)}(t)$ for all $i \in \mathbb{Z}$ and all $t \geq 0$, almost surely.

We take as auxiliary processes $z^{(k)}(t)$ suitable translations of the basic wedge process defined in (13)–(14). For $k \in \mathbb{Z}$ set

$$w_i^{(k)}(0) = \begin{cases} i - k, & i < k \\ 0, & i \geq k \end{cases} \quad \text{and} \quad z_i^{(k)}(t) = h_k(0) + w_i^{(k)}(t). \quad (26)$$

The apex of the wedge $z^{(k)}(0)$ is at the point $(k, h_k(0))$, and then the definition of the wedge ensures that $h(0) \geq z^{(k)}(0)$. Hypothesis (25) holds and Lemma 2.1 gives this variational equality:

$$h_i(t) = \sup_{k \in \mathbb{Z}} \{h_k(0) + w_i^{(k)}(t)\}. \quad (27)$$

Now we extract two results from the assembled ingredients: the function g of (18), and a general “hydrodynamic limit” that describes the large scale evolution of the process.

First specialize (27) to the stationary situation where $h_0(0) = 0$ and the increments are ν_ρ -distributed, and write (27) in the form

$$t^{-1}h_0(t) = \sup_{y \in \mathbb{R}} \{t^{-1}h_{[ty]}(0) + t^{-1}w_0^{([ty])}(t)\}. \quad (28)$$

Let $t \rightarrow \infty$. Inside the braces on the right $t^{-1}h_{[ty]}(0) \rightarrow \rho y$ a.s. by the law of large numbers. $t^{-1}w_0^{([ty])}(t) \rightarrow g(-y)$ by a translation of the limit (18). With some work take the limit outside the supremum. Then we know $t^{-1}h_0(t)$ converges. By supplying some moment bounds we can take expectations over the limits, and with (23) arrive at

$$-f(\rho) = \sup_{y \in \mathbb{R}} \{\rho y + g(-y)\}. \quad (29)$$

This is a convex duality (Rockafellar, 1970). From the explicit invariant distributions (21) we obtained f in (24), and then we can solve (29) for g . (Without the invariant distributions we can carry out part of this reasoning but we cannot find f and g explicitly.) Let us record the results.

Theorem 2.2 *For the limit (18) $g(x) = -\frac{1}{4}(1-x)^2$ for $-1 \leq x \leq 1$. For the limit (9) $\gamma(x, y) = (\sqrt{x} + \sqrt{y})^2$ for $x, y \geq 0$.*

We turn to the hydrodynamic limit. Assume given a function u_0 on \mathbb{R} and a sequence of random initial height functions $h^n(0) \in \mathcal{X}_1$ ($n \in \mathbb{N}$) such that

$$n^{-1}h_{[nx]}^n(0) \rightarrow u_0(x) \quad \text{a.s. as } n \rightarrow \infty \text{ for each } x \in \mathbb{R}. \quad (30)$$

For this to be possible u_0 has to be Lipschitz with $0 \leq u_0'(x) \leq 1$ Lebesgue-a.e.

Theorem 2.3 *For $x \in \mathbb{R}$ and $t > 0$ we have the limit*

$$n^{-1}h_{[nx]}^n(nt) \rightarrow u(t, x) \quad \text{a.s. as } n \rightarrow \infty \quad (31)$$

where

$$u(t, x) = \sup_{y \in \mathbb{R}} \left\{ u_0(y) + tg\left(\frac{x-y}{t}\right) \right\}. \quad (32)$$

Equation (32) is a Hopf-Lax formula (Evans, 1998) and it says that u is the entropy solution of the Hamilton-Jacobi equation

$$u_t + f(u_x) = 0, \quad u|_{t=0} = u_0. \quad (33)$$

In other words this equation governs the macroscopic evolution of the height process. Theorem 2.3 is proved by showing that, as $n \rightarrow \infty$, variational formula (27) for $n^{-1}h_{[nx]}^n(nt)$ turns into (32). Details can be found in (Seppäläinen, 1999).

Further remarks. The function g in Theorem 2.2 was first calculated by Rost (1981) in one of the seminal papers of hydrodynamic limits, but without the last-passage representation and with a different approach than the one presented here.

Let us discuss various avenues of generalization. We encounter immediately difficult open problems.

(i) *Generalizations that retain the envelope property.* The argument sketched above that combines the envelope property with the duality of the flux and the wedge shape to derive hydrodynamic limits was introduced in (Seppäläinen, 1998a,c, 1999). An earlier instance of the variational connection appeared in Aldous and Diaconis (1995) for Hammersley’s process. This work itself was based on the classic paper (Hammersley, 1972); see Section 3.2 below. Also in queueing literature similar variational expressions arise (Szczotka and Kelly, 1990).

To define the K -exclusion process we replace the state space \mathcal{X}_1 of (15) with

$$\mathcal{X}_K = \{h \in \mathbb{Z}^{\mathbb{Z}} : 0 \leq h_i - h_{i-1} \leq K \text{ for all } i \in \mathbb{Z}\} \quad (34)$$

for some $2 \leq K < \infty$. Otherwise keep the model the same: rate 1 Poisson clocks $\{N_i\}$ govern the jumps of height variables h_i , and jumps that take the state h outside the space \mathcal{X}_K are prohibited. The increment process is now called totally asymmetric K -exclusion (some authors use “generalized exclusion”). The variational coupling (Lemma 2.1) works as before. But invariant distributions are unknown, and there is even no proof of existence of an extremal invariant distribution for each density value $\rho \in [0, K]$. No alternative way to compute f and g has been found. Theorem 2.3 is valid, but the most that can be said about f and g is that they exist as concave functions.

Interestingly, the situation becomes again explicitly analyzable for $K = \infty$ where the only constraint on h is $h_i \leq h_{i+1}$. The increment process is a special case of a *zero range process*. Its state space is $(\mathbb{Z}_+)^{\mathbb{Z}}$ and IID geometric distributions are invariant (Liggett, 1973; Andjel, 1982). As a final step of generalization, away from monotone height functions, let us mention *bricklayer processes* (Balázs, 2003; Balázs et al., 2007) whose increments $\eta_i = h_i - h_{i-1}$ can be positive or negative.

The variational coupling of Lemma 2.1 works equally well for certain multidimensional height processes $h(t) = (h_i(t) : i \in \mathbb{Z}^d)$ of the type discussed here. Examples appear in (Rezakhanlou, 2002b; Seppäläinen, 2000, 2007). No explicit invariant distributions are known for multidimensional height models. The variational scheme proves that scaled height processes converge to solutions of Hamilton-Jacobi equations as in Theorem 2.3. But again one can only assert the existence of f and g instead of giving them explicitly.

Another direction of generalization is to let the weights $\{Y_{i,j}\}$ have distributions other than exponential or geometric. The height process $h(t)$ ceases to be Markovian but the last-passage model of Figs. 1 and 2 makes sense. As mentioned, the limit $\gamma(x, y)$ in (9) is explicitly known only for the exponential and geometric cases. A distribution as simple as Bernoulli ($Y_{i,j}$ takes only two

values) cannot be handled. However, if the paths are altered to require that one or both coordinates increase strictly, then the variational approach does find the explicit shape for the Bernoulli case (Seppäläinen, 1997, 1998b).

Thus the present situation is that an explicit limit shape can be found only for some fortuitous combinations of path geometries and weight distributions.

(ii) *Partially asymmetric models.* Let us next address the case where the column variables h_i are allowed to jump both up and down. Fix two parameters $0 < q < p$ such that $p + q = 1$ (convenient normalization). Give each column i two independent Poisson clocks, $N_i^{(-)}$ with rate p and $N_i^{(+)}$ with rate q . At jump times of $N_i^{(\pm)}$ variable h_i attempts to jump to $h_i \pm 1$, and as before, a jump is completed if its execution does not take the state out of the state space \mathcal{X}_1 . For the increment process this means that a 10 pair becomes a 01 pair at rate p , and the opposite move happens at rate q . Bernoulli distributions (21) are still invariant. This increment process is the *asymmetric simple exclusion process* (ASEP). In the same vein one can allow K particles per site and talk about asymmetric K -exclusion.

The envelope property of Lemma 2.1 is now lost. An alternative approach from (Rezakhanlou, 2001) utilizes compactness of the random semigroups of the height process. Limit points are characterized as Hamilton-Jacobi semigroups via the Lions-Nisio theorem (Lions and Nisio, 1982). Thereby Theorem 2.3 is derived for one-dimensional asymmetric K -exclusion. For $K = 1$ the flux f in (33) must be replaced by $f(\rho) = (p - q)\rho(1 - \rho)$, while for $2 \leq K < \infty$ the flux is unknown. In the multidimensional case it is not known if the resulting equation itself is random or not.

3 Fluctuations

A simple way to create initial height functions $h^n(0)$ that satisfy assumption (30) is to take independent increments with distributions

$$\mathbf{P}[\eta_i^n(0) = 1] = n(u_0(\frac{i}{n}) - u_0(\frac{i-1}{n})),$$

and at the origin assign the deterministic value $h_0^n(0) = [nu_0(0)]$. The stationary situation is of this type with $u_0(x) = \rho x$. Then initial fluctuations

$$n^{-1/2}\{h_{[nx]}^n(0) - nu_0(x)\}$$

are Gaussian in the limit $n \rightarrow \infty$. This makes it natural to look for a distributional limit at later times $t > 0$ on the central limit scale $n^{1/2}$:

$$n^{-1/2}\{h_{[nx]}^n(nt) - nu(x, t)\} \longrightarrow \zeta(t, x) \quad \text{as } n \rightarrow \infty, \quad (35)$$

for some limit process $\zeta(t, x)$. Such limits can be proved but process $\{\zeta(t, x)\}$ turns out to be a deterministic function of the initial fluctuations $\{\zeta(0, x)\}$. Consequently limit (35) does not record any fluctuations created by the dynamics. Theorem 3.3 below gives a precise statement of this type.

In asymmetric systems the fluctuations created by the dynamics occur on a scale smaller than $n^{1/2}$. Two types of such phenomena have been found. Processes related to the last-passage model and exclusion process discussed in Section 2 have order $n^{1/3}$ fluctuations whose limits are distributions from random matrix theory. A class of linear processes has order $n^{1/4}$ fluctuations and Gaussian limits related to fractional Brownian motion with Hurst parameter $H = 1/4$. To see these lower order fluctuations one can start the system with a deterministic initial state, or one can start the system in the stationary distribution or some other random state but then follow the evolution along characteristic curves of the macroscopic PDE. The fluctuation situation is very different for symmetric systems; the reader can consult (Kipnis and Landim, 1999, Chapter 11).

3.1 Exclusion process

Probability distributions from random matrix theory were discovered as limit laws for last-passage growth models almost a decade ago.

Theorem 3.1 (Johansson, 2000). *For the corner growth model*

$$\mathbf{P} \left[\frac{G([xn], [ny]) - n\gamma(x, y)}{c(x, y)n^{1/3}} \leq s \right] \longrightarrow F(s) \quad \text{as } n \rightarrow \infty, \quad (36)$$

where F is the Tracy-Widom GUE distribution.

The distribution F first appeared as the limit distribution of the scaled largest eigenvalue of a random Hermitian matrix from the GUE (Tracy and Widom, 1994). GUE is short for *Gaussian Unitary Ensemble*. This means that a random Hermitian matrix is constructed by putting IID complex-valued Gaussian random variables above the diagonal, IID real-valued Gaussian random variables on the diagonal, and letting the Hermitian property determine the entries below the diagonal. Then as the matrix grows in size, the variances of the entries are scaled appropriately to obtain limits. The standard reference is (Mehta, 2004).

Theorem 3.1 and related results initially arose entirely outside probability theory (except for the statements themselves), involving the RSK correspondence and Gessel's identity from combinatorics and techniques from integrable systems to analyze the asymptotics of the resulting determinants. The *RSK correspondence*, named after Robinson, Schensted and Knuth, is a bijective mapping between certain arrays of integers or integer matrices (in this case the matrix in Figure 1 if the $Y_{i,j}$'s are integers) and pairs of Young tableaux. These latter objects are ubiquitous in combinatorics. Standard references are (Fulton, 1997; Sagan, 2001). More recently determinantal point processes have appeared as the link between the growth processes and random matrix theory. We shall not pursue these topics further for many excellent reviews are available: Baik (2005), Deift (2000), Johansson (2002), König (2005) and Spohn (2006).

Precise limits such as (36) have so far been restricted to totally asymmetric systems. Next we discuss ideas that fall short of exact limits but do give the correct order of the variance of the height for partially asymmetric systems.

Consider the height process $h(t)$ whose increments $\eta_i(t) = h_i(t) - h_{i-1}(t)$ form the asymmetric simple exclusion process (ASEP). This process was introduced in the remarks at the end of Section 2. Each height variable h_i attempts downward jumps with rate p and upward jumps with rate q , and $p > q$. A jump is suppressed if it would lead to a violation of the restrictions $h_{i-1} \leq h_i$ or $h_i \geq h_{i+1} - 1$ encoded in the state space \mathcal{X}_1 of (15). In the increment process each 10 pair becomes a 01 pair at rate p and each 01 pair becomes 10 pair at rate q .

On large space and time scales the height process obeys the Hamilton-Jacobi equation

$$u_t + f(u_x) = 0 \text{ with } f(\rho) = (p - q)\rho(1 - \rho).$$

This PDE carries information along the curves $\dot{x} = f'(u_x(t, x))$, in the sense that the slope u_x is constant along these curves as long as it is continuous. At constant slope $u_x = \rho$ the characteristic speed is $V^\rho = f'(\rho) = (p - q)(1 - 2\rho)$.

Consider the stationary process: $0 < \rho < 1$ is fixed, and at each time $t \geq 0$ the increments $\{\eta_i(t)\}_{i \in \mathbb{Z}}$ have Bernoulli ν_ρ -distribution from (21). Normalize the heights by setting initially $h_0(0) = 0$. We determine the order of magnitude of the variance of the height as seen by an observer traveling at speed V^ρ .

Theorem 3.2 (Balázs and Seppäläinen, 2007b) *Height fluctuations along the characteristic satisfy*

$$0 < \liminf_{t \rightarrow \infty} t^{-2/3} \mathbf{Var}\{h_{[V^\rho t]}(t)\} \leq \limsup_{t \rightarrow \infty} t^{-2/3} \mathbf{Var}\{h_{[V^\rho t]}(t)\} < \infty.$$

If the observer choses any other speed $v \neq V^\rho$, only a translation of initial Gaussian fluctuations would be observed. Take $v > V^\rho$ to be specific. Due to the normalization $h_0(0) = 0$ we can write

$$h_{[vt]}(t) = (h_{[vt]}(t) - h_{[(v-V^\rho)t]}(0)) + \sum_{i=1}^{[(v-V^\rho)t]} \eta_i(0). \quad (37)$$

On the right the first expression in parentheses is a height increment along a characteristic and so by the theorem has fluctuations of order $t^{1/3}$. The last sum of initial increments has Gaussian fluctuations of order $t^{1/2}$ and consequently drowns out the first term.

The proof of Theorem 3.2 is entirely different from the proofs of Theorem 3.1. As it involves an important probabilistic idea let us discuss it briefly.

Couplings enable us to study the evolution of discrepancies between processes. In exclusion processes these discrepancies are called second class particles. Consider two initial ASEP configurations $\eta(0), \zeta(0) \in \{0, 1\}^{\mathbb{Z}}$. The configurations differ at the origin: $\zeta(0)$ has a particle at the origin ($\zeta_0(0) = 1$) but $\eta(0)$ does not ($\eta_0(0) = 0$). At all other sites $i \neq 0$ we give the configurations a common but random value $\eta_i(0) = \zeta_i(0)$ according to the mean ρ Bernoulli distribution. Let the joint process $(\eta(t), \zeta(t) : t \geq 0)$ evolve together governed

by the *same* Poisson clocks. The effect of this coupling is that there is always exactly one site $Q(t)$ such that $\zeta_{Q(t)}(t) = 1$, $\eta_{Q(t)}(t) = 0$, and $\eta_i(t) = \zeta_i(t)$ for all $i \neq Q(t)$.

$Q(t)$ is the location of a *second class particle* relative to the process $\eta(t)$. (Relative to $\zeta(t)$ one should say “second class *antiparticle*.”) In addition to ordinary exclusion jumps, Q yields to η -particles: if an η -particle at $Q + 1$ jumps left (rate q) then Q jumps right to switch places with the η -particle. Similarly an η -particle at $Q - 1$ switches places with Q at rate p . These special jumps follow from considering the effects of clocks $N_{Q\pm 1}^{(\mp)}$ on the discrepancy between η and ζ .

Proof of Theorem 3.2 utilizes couplings of several processes with different initial conditions. Evolution of second class particles is directly related to differences in particle current (height) between processes. On the other hand Q and the height variance are related through this identity:

$$\mathbf{Var}\{h_{[vt]}(t)\} = \rho(1 - \rho)\mathbf{E}\{|Q(t) - [vt]| \} \quad \text{for any } v. \quad (38)$$

The right-hand side can be expected to have order smaller than t precisely when $v = V^\rho$ on account of this second identity:

$$\mathbf{E}Q(t) = tV^\rho. \quad (39)$$

From these ingredients the bounds in Theorem 3.2 arise.

Further remarks. As already suggested at the end of Section 2, a major problem for growth models is to find robust techniques that are not dependent on particular choices of probability distributions or path geometries. Progress on fluctuations of the corner growth model beyond the exponential case has come in situations that are in some sense extreme: for distributions with heavy tails (Hambly and Martin, 2007) or for points close to the boundary of the quadrant (Baik and Suidan, 2005; Bodineau and Martin, 2005). See review by Martin (2006).

The second class particle appears in many places in interacting particle systems. In the hydrodynamic limit picture second class particles converge to characteristics and shocks of the macroscopic PDE (Ferrari and Fontes, 1994b; Rezakhanlou, 1995; Seppäläinen, 2001). Versions of identities (38) and (39) are valid for zero range and bricklayer processes (Balázs and Seppäläinen, 2007a). Equation (39) is surprising because the process as seen by the second class particle is *not* stationary.

In general the view of the process from the second class particle is complicated. Studies of invariant distributions seen by second class particles appear in (Derrida et al., 1993; Ferrari et al., 1994; Ferrari and Martin, 2007). There are special cases of parameter values for certain processes where unexpected simplification takes place and the process seen from the second class particle has a product-form invariant distribution (Derrida et al., 1997; Balázs, 2001).

3.2 Hammersley process

We began this paper with the exclusion process because this process is by far the most studied among its kind. It behooves us to introduce also Hammersley's process for which several important results were proved first. It has an elegant graphical construction that is related to a classical combinatorial question, namely the maximal length of an increasing subsequence of a random permutation. This goes back to (Hammersley, 1972); see also (Aldous and Diaconis, 1995, 1999).

We begin with the growth model. Put a homogeneous rate 1 Poisson point process on the plane. This is a random discrete subset of the plane characterized by the following property: the number of points in a Borel set B is Poisson distributed with mean given by the area of B and independent of the points outside B . Call a sequence $(x_1, t_1), (x_2, t_2), \dots, (x_k, t_k)$ of these Poisson points *increasing* if $x_1 < x_2 < \dots < x_k$ and $t_1 < t_2 < \dots < t_k$. Let $L((a, s), (b, t))$ be the maximal number of points on an increasing sequence in the rectangle $(a, b] \times (s, t]$ (Fig. 6). The random permutation comes from mapping the ordered x -coordinates to ordered t -coordinates in the rectangle, and $L((a, s), (b, t))$ is precisely the maximal length of an increasing subsequence of this permutation.

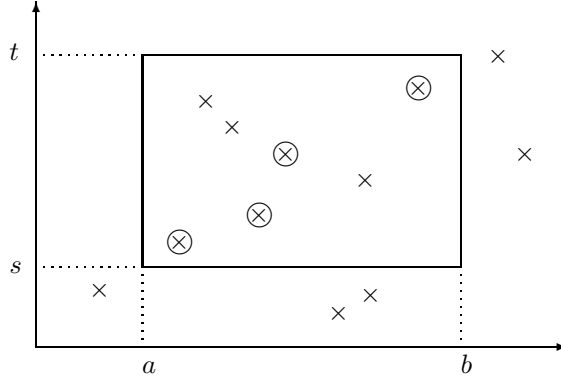


Figure 6: Increasing sequences among planar Poisson points marked by \times 's. $L((a, s), (b, t)) = 4$ as shown by the circled Poisson points that form an increasing sequence.

The limit

$$\lim_{n \rightarrow \infty} n^{-1} L((0, 0), (nx, nt)) = 2\sqrt{xt} \quad \text{a.s.} \quad (40)$$

holds for $x, t > 0$. The limit exists by superadditivity exactly as for (9). The functional form $c\sqrt{xt}$ follows from scaling properties of the Poisson process. The value $c = 2$ was first derived by Veršik and Kerov (1977), while Logan and Shepp (1977) independently proved $c \geq 2$. The fluctuation result for L is analogous to (36), with normalization $n^{1/3}$ and the Tracy-Widom limit (Baik et al., 1999).

We embed the increasing sequences in the graphical construction of the *Hammersley process*. This process consists of point particles that move on \mathbb{R} by jumping. Put a rate 1 Poisson point process on the space-time plane and place the particles initially on the real axis. Move the real axis up at constant speed 1. Each Poisson point (x, t) instantaneously pulls to x the next particle to the right of x . We label the particles from left to right: $z_i(t) \in \mathbb{R}$ is the position of particle i at time t . We could regard the variables z_i as heights again, but the particle picture seems more compelling. This construction is illustrated by Fig. 7. In terms of infinitesimal rates, the construction realizes this rule: independently of other particles, at rate $z_i - z_{i-1}$ variable z_i jumps to a uniformly chosen location in the interval (z_{i-1}, z_i) .

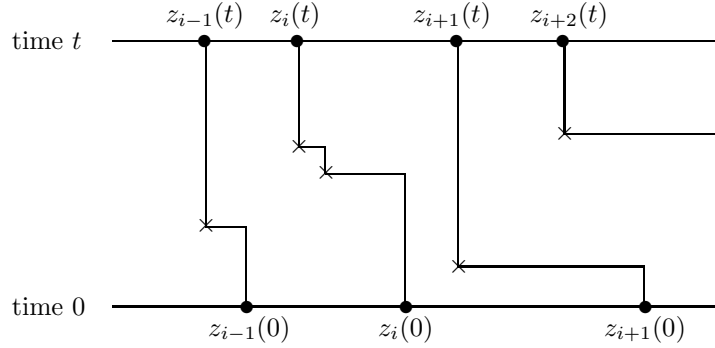


Figure 7: Portion of the graphical construction of Hammersley's process. \times 's mark space-time Poisson points. \bullet 's mark particle locations at time 0 and at a later time $t > 0$. Space-time trajectories of particles are shown. The horizontal segments are traversed instantaneously and the vertical segments at constant speed 1.

As in Section 2, there is a variational characterization for this construction. Define an inverse for the maximal path variable $L((a, s), (b, t))$ by

$$\Gamma((a, s), t, w) = \inf\{h \geq 0 : L((a, s), (a + h, t)) \geq w\}.$$

Take an initial particle configuration $\{z_i(0)\} \in \mathbb{R}^{\mathbb{Z}}$ that satisfies $z_{i-1}(0) \leq z_i(0)$ and $i^{-2}z_i(0) \rightarrow 0$ as $i \rightarrow -\infty$. Then the graphical construction leads to a well-defined evolution $\{z_i(t)\}$ that satisfies

$$z_i(t) = \inf_{k:k \leq i} \{z_k(0) + \Gamma((z_k(0), 0), t, i - k)\}. \quad (41)$$

Here is the hydrodynamic limit. Consider a sequence of processes $z^n(t)$ that satisfies $n^{-1}z^n_{[ny]}(0) \rightarrow u_0(y)$ for each $y \in \mathbb{R}$, say in probability. The initial function u_0 is nondecreasing, locally Lipschitz and satisfies

$$y^{-2}u_0(y) \rightarrow 0 \quad \text{as } y \rightarrow -\infty. \quad (42)$$

Define

$$u(t, x) = \inf_{y: y \leq x} \left\{ u_0(y) + \frac{(x - y)^2}{4t} \right\}, \quad (t, x) \in (0, \infty) \times \mathbb{R}. \quad (43)$$

Since rates are unbounded, we need to assume a left tail bound to prevent the particles from disappearing to $-\infty$: given $\varepsilon > 0$ there exist $0 < q, n_0 < \infty$ such that

$$\mathbf{P}\{z_i^n(0) < -\varepsilon i^2/n \text{ for some } i \leq -qn\} \leq \varepsilon \quad \text{for } n \geq n_0.$$

Under these assumptions

$$n^{-1} z_{[nx]}^n(nt) \rightarrow u(t, x) \quad \text{in probability} \quad (44)$$

(Seppäläinen, 1996). The function defined by the Hopf-Lax formula (43) solves the Hamilton-Jacobi equation $u_t + (u_x)^2 = 0$.

Let us state a precise result about the central limit scale fluctuations (35) that covers also shocks. For $(t, x) \in (0, \infty) \times \mathbb{R}$ let

$$I(t, x) = \left\{ y \in (-\infty, x] : u(t, x) = u_0(y) + \frac{(x - y)^2}{4t} \right\} \quad (45)$$

be the set of minimizers in (43), guaranteed nonempty and compact by hypothesis (42). Then (t, x) is a *shock* if $I(t, x)$ is not a singleton. This is equivalent to the nonexistence of the x -derivative $u_x(t, x)$.

Fluctuations on the scale $n^{1/2}$ from the limit (44) are described by the process

$$\zeta_n(t, x) = n^{-1/2} \{z_{[nx]}^n(nt) - nu(x, t)\}.$$

Assume the existence of a continuous random function ζ_0 on \mathbb{R} such that the convergence in distribution $\zeta_n(0, \cdot) \rightarrow \zeta_0$ holds in the topology of uniform convergence on compact sets. Define the process ζ by

$$\zeta(t, x) = \inf_{y \in I(t, x)} \zeta_0(y)$$

where $I(t, x)$ is the (deterministic) set defined in (45).

Theorem 3.3 *For each (t, x) , $\zeta_n(t, x) \rightarrow \zeta(t, x)$ in distribution.*

As stated in the beginning of Section 3, this distributional limit reflects no contribution from dynamical fluctuations as the process ζ is a deterministic transformation of ζ_0 . The underlying reason is that the dynamical fluctuations of order $n^{1/3}$ are not visible on the $n^{1/2}$ scale. The dynamical fluctuations are the universal ones described by the Tracy-Widom laws. See again the discussion and references that follow Theorem 3.1.

Further remarks. The *polynuclear growth model* (PNG) is another related (1+1)-dimensional growth model used by several authors for studies of Tracy-Widom fluctuations and the Airy process in the KPZ scaling picture (Baik and Rains, 2000; Ferrari, 2004; Johansson, 2003; Prähofer and Spohn, 2002, 2004). Like the Hammersley process, the graphical construction of the

PNG utilizes a planar Poisson process, and in fact the same underlying last-passage model of increasing paths. This time the Poisson points mark space-time nucleation events from which new layers grow laterally at a fixed speed. Roughly speaking, this corresponds to putting the time axis at a 45 degree angle in Figs. 6 and 7.

More about the phenomena related to Theorem 3.3 can be found in article (Seppäläinen, 2002). A similar theorem for TASEP appears in (Rezakhanlou, 2002a). Earlier work on the diffusive fluctuations of ASEP was done by Ferrari and Fontes (1994a,b).

3.3 Linear models

We turn to systems macroscopically governed by linear first order equations $u_t + bu_x = 0$. Fluctuations across the characteristic occur now on the scale $n^{1/4}$ and converge to a Gaussian process related to fractional Brownian motion.

The *random average process* (RAP) was first studied by Ferrari and Fontes (1998). The state of the process is a height function $\sigma : \mathbb{Z} \rightarrow \mathbb{R}$ with $\sigma_i \in \mathbb{R}$ denoting the height over site i . (More generally the domain can be \mathbb{Z}^d .) The basic step of the evolution is that a value σ_i is replaced by a weighted average of values in a neighborhood, and the randomness comes in the weights. This time we consider a discrete time process. The basic step is carried out simultaneously at all sites i .

Now precise formulations. Let $\{u(k, \tau) : k \in \mathbb{Z}, \tau \in \mathbb{N}\}$ be an IID collection of random probability vectors indexed by space-time $\mathbb{Z} \times \mathbb{N}$. In terms of coordinates $u(k, \tau) = (u_j(k, \tau) : -M \leq j \leq M)$. We assume the system has finite range defined by the fixed parameter M . We impose a minimal assumption that guarantees that the weight vectors are not entirely degenerate:

$$\mathbf{P}\{\max_j u_j(0, 0) < 1\} > 0. \quad (46)$$

For technical convenience we also assume that the process is “on the correct lattice”: there does not exist an integer $h \geq 2$ such that for some $b \in \mathbb{Z}$ the mean weights $p(j) = \mathbf{E}u_j(0, 0)$ satisfy $\sum_{j \in b+h\mathbb{Z}} p(j) = 1$.

To start the dynamics let $\sigma(0)$ be a given random or deterministic initial height function. The process $\sigma(\tau)$, $\tau = 0, 1, 2, \dots$, is defined iteratively by

$$\sigma_i(\tau) = \sum_j u_j(i, \tau) \sigma_{i+j}(\tau - 1), \quad \tau \geq 1, i \in \mathbb{Z}. \quad (47)$$

As before, we can define the process of increments $\eta_i(\tau) = \sigma_i(\tau) - \sigma_{i-1}(\tau)$. The increments also evolve via random linear mappings and are conserved like particles in exclusion processes.

As in Section 2 we create suitable initial conditions for a hydrodynamic limit. Consider a sequence of processes $\sigma^n(\tau)$ indexed by $n \in \mathbb{N}$, initially normalized by $\sigma_0^n(0) = 0$. For each n assume independent initial increments $\{\eta_i^n(0) : i \in \mathbb{Z}\}$ with

$$\mathbf{E}[\eta_i^n(0)] = \rho(i/n) \quad \text{and} \quad \mathbf{Var}[\eta_i^n(0)] = v(i/n) \quad (48)$$

for given Hölder $1/2 + \varepsilon$ functions ρ and v . Assume a uniform moment bound: $\sup_{n,i} \mathbf{E}[|\eta_i^n(0)|^{2+\delta}] < \infty$ for some $\delta > 0$.

The hydrodynamic limit is rather trivial for it consists only of translation. Define a function u on \mathbb{R} by $u(0) = 0$ and $u'(x) = \rho(x)$. The characteristic speed is

$$b = - \sum_j j p(j).$$

Then for each $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$,

$$n^{-1} \sigma_{[nx]}^n([nt]) \longrightarrow u(x - bt) \quad \text{as } n \rightarrow \infty, \text{ in probability.}$$

In other words, the height obeys the linear PDE $u_t + b u_x = 0$.

On the central limit scale $n^{1/2}$ one would also see only translation of initial fluctuations. To see something nontrivial we look at fluctuations around a characteristic line. Fix a point $\bar{y} \in \mathbb{R}$ and consider the characteristic line $t \mapsto \bar{y} + tb$ emanating from $(\bar{y}, 0)$. Define space-time process

$$Z_n(t, r) = \sigma_{[n\bar{y}] + [r\sqrt{n}] + [ntb]}^n([nt]) - \sigma_{[n\bar{y}] + [r\sqrt{n}]}^n(0), \quad (t, r) \in \mathbb{R}_+ \times \mathbb{R}.$$

The spatial variable r describes fluctuations around the characteristic on the spatial scale $n^{1/2}$. For the increment process $Z_n(t, 0)$ represents net current from right to left across the characteristic.

Theorem 3.4 (Balázs et al., 2006) *The finite-dimensional distributions of the process $n^{-1/4} Z_n$ converge to those of the Gaussian process $\{z(t, r) : t \geq 0, r \in \mathbb{R}\}$ described below.*

The statement means that for any finite collection of space-time points $(t_1, r_1), \dots, (t_k, r_k)$, the \mathbb{R}^k -valued random vector $n^{-1/4}(Z_n(t_1, r_1), \dots, Z_n(t_k, r_k))$ converges in distribution to the vector $(z(t_1, r_1), \dots, z(t_k, r_k))$. The limiting process z has the following representation in terms of stochastic integrals:

$$\begin{aligned} z(t, r) = & \rho(\bar{y}) \sigma_a \sqrt{\kappa} \iint_{[0, t] \times \mathbb{R}} \varphi_{\sigma_a^2(t-s)}(r - z) dW(s, z) \\ & + \sqrt{v(\bar{y})} \int_{\mathbb{R}} \text{sign}(x - r) \Phi_{\sigma_a^2 t}(-|x - r|) dB(x). \end{aligned} \quad (49)$$

Above W is a 2-parameter Brownian motion on $\mathbb{R}_+ \times \mathbb{R}$ and B is a 1-parameter Brownian motion on \mathbb{R} independent of W . The first integral represents dynamical noise generated by the random weights, and the second the initial noise propagated by the evolution. The functions in the integrals are Gaussian densities and distribution functions:

$$\varphi_{\sigma^2}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{x^2}{2\sigma^2}\right\} \quad \text{and} \quad \Phi_{\sigma^2}(x) = \int_{-\infty}^x \varphi_{\sigma^2}(y) dy.$$

The only effects from the initial height are the mean $\rho(\bar{y})$ and variance $v(\bar{y})$ of the increments around the point $n\bar{y}$. The parameter σ_a^2 is the variance of the probabilities $p(j)$, and κ another parameter determined by the distribution of the weights. Process z has a self-similarity property: $\{z(at, a^{1/2}r)\} \stackrel{d}{=} \{a^{1/4}z(t, r)\}$.

In the special case where $v(\bar{y}) = \kappa\rho(\bar{y})^2$ the temporal process $\{z(t, r) : t \in \mathbb{R}_+\}$ (for any fixed r) has covariance

$$\mathbf{E}z(s, r)z(t, r) = \frac{\sigma_a \kappa \rho^2}{\sqrt{2\pi}} (\sqrt{s} + \sqrt{t} - \sqrt{|t-s|}).$$

This identifies $z(\cdot, r)$ as *fractional Brownian motion* with Hurst parameter $H = 1/4$. In particular, this limit arises in a stationary case where the averaging involves two points and the weight is beta distributed (Balázs et al., 2006, Example 2.1).

The proof of Theorem 3.4 utilizes a special case of another stochastic model of great contemporary interest, namely random walk in random environment (RWRE). Here is how the RWRE arises. An environment $\omega = \{u(k, \tau)\}$ is determined by the weight vectors. Given ω , define a “backward” walk $\{X_s^{i, \tau} : s \in \mathbb{Z}_+\}$ on \mathbb{Z} with initial position $X_0^{i, \tau} = i$ and transition probability

$$P^\omega(X_{s+1}^{i, \tau} = y | X_s^{i, \tau} = x) = u_{y-x}(x, \tau - s), \quad s = 0, 1, 2, \dots$$

The superscript ω on P^ω indicates that it is the *quenched* path measure of $\{X_s^{i, \tau} : s \in \mathbb{Z}_+\}$ under a fixed ω . The basic step (47) of RAP evolution can be rewritten so that $\sigma_i(\tau)$ equals the average value of the previous height function $\sigma(\tau - 1)$ seen by a walk started at i after one step:

$$\sigma_i(\tau) = \sum_j u_{j-i}(i, \tau) \sigma_j(\tau - 1) = E^\omega[\sigma_{X_1^{i, \tau}}(\tau - 1)].$$

This can be iterated all the way down to the initial height function:

$$\sigma_i(\tau) = E^\omega[\sigma_{X_\tau^{i, \tau}}(0)].$$

Note that the expectation E^ω over paths of the walk $X_s^{i, \tau}$ under fixed weights ω sees the initial height function $\{\sigma_i(0)\}_{i \in \mathbb{Z}}$ as a constant.

We have here a special type of RWRE called “space-time.” Another term used is “dynamical environment” because after each step the walk sees a new sample of its environment. Proof of Theorem 3.4 requires limits for the walk itself and its quenched mean process $E^\omega(X_s^{i, \tau})$. These results appear in (Balázs et al., 2006; Rassoul-Agha and Seppäläinen, 2005).

Independent walks on \mathbb{Z} display the same behavior as RAP. Let the process $Z_n(t, r)$ be the net particle current across the characteristic $t \mapsto [n\bar{y}] + [r\sqrt{n}] + [ntb]$ where b is the common average speed of the particles. Then under suitable assumptions on the initial particle arrangements and their jump kernel, Z_n satisfies a stronger form of Theorem 3.4 that also contains process-level convergence. One adjustment is necessary: the constants in front of the stochastic integrals in (49) are different for the random walk case. The stationary system sees again fractional Brownian motion as the limit of the current. Details for the random walk case appear in (Seppäläinen, 2005; Kumar, 2007). Earlier related results for a Poisson system of independent Brownian motions appeared in (Dürr et al., 1985).

4 Large deviations

We present the large deviation picture for the Hammersley process, so we continue in the setting of Section 3.2. Recall the definition of the longest path model among planar Poisson points illustrated in Fig. 6. Abbreviate $L_n = L((0, 0), (n, n))$. Then the limit (40) is $n^{-1}L_n \rightarrow 2$. Here is the large deviation theorem for L_n . It was completed shortly before the fluctuation result of Baik et al. (1999), through a combination of several independent papers: Logan and Shepp (1977), Kim (1996), Seppäläinen (1998d) and Deuschel and Zeitouni (1999).

Theorem 4.1 *We have the following upper and lower tail large deviation bounds.*

$$\lim_{n \rightarrow \infty} n^{-1} \log \mathbf{P}\{L_n \geq nx\} = -I(x) \quad \text{for } x \geq 2 \quad (50)$$

with rate function $I(x) = 2x \cosh^{-1}(x/2) - 2\sqrt{x^2 - 4}$.

$$\lim_{n \rightarrow \infty} n^{-2} \log \mathbf{P}\{L_n \leq nx\} = -U(x) \quad \text{for } 0 \leq x \leq 2 \quad (51)$$

with rate function $U(x) = \int_x^2 R_2(s) ds$ where $R_2(s) = s \log(s/2) - s + 2$ is the rate function for IID mean 2 Poisson random variables.

To develop this theme further we state a lower tail LDP for the tagged particle in Hammersley's process. An interesting feature is that the large deviation rate functions again obey the Hopf-Lax semigroup formula, as did the limit (44). The assumption is that lower tail rate functions exist initially: for all $y, s \in \mathbb{R}$ the limit

$$J_0(y, s) = - \lim_{n \rightarrow \infty} n^{-1} \log \mathbf{P}\{z_{[ny]}^n(0) \leq ns\}$$

exists and is left continuous in y for each fixed s . Define

$$\Psi(w, r) = - \lim_{n \rightarrow \infty} n^{-1} \log \mathbf{P}\{\Gamma((0, 0), (n, nw)) \leq nr\}. \quad (52)$$

This limit exists by superadditivity. For technical reasons a uniform tail bound is needed for the initial particle locations: there exist constants $0 < C_j < \infty$ such that

$$\mathbf{P}\{z_i^n(0) \leq -C_1|i|\} \leq e^{-C_2|i|} \quad \text{for } i \leq -C_3n, \text{ for large enough } n.$$

Theorem 4.2 (Seppäläinen, 1998d) *The limit*

$$J_t(x, r) = - \lim_{n \rightarrow \infty} n^{-1} \log \mathbf{P}\{z_{[nx]}^n(nt) \leq nr\}$$

exists for all $x, r \in \mathbb{R}$ and $t > 0$, and is given by

$$J_t(x, r) = \inf_{(y, s): y \leq x, s \leq r} \left\{ J_0(y, s) + t\Psi\left(\frac{x-y}{t}, \frac{r-s}{t}\right) \right\}. \quad (53)$$

The approach of Section 2 can be adapted to prove the upper tail LDP (50) and Theorem 4.2. The stationary systems make explicit calculation again possible. Presently it is not clear how to include the lower tail LDP (51) in the variational framework. Hence this part requires a separate proof. Details appear in Deuschel and Zeitouni (1999) and Seppäläinen (1998d).

Further remarks. Even though Theorem 4.1 has explicit rate functions, this large deviation problem remains unfinished in an important sense. It is not understood how the system behaves to create a deviation, and it is not clear what the rate functions I and U represent. The present proofs are too indirect.

Let us illustrate through the random walk LDP (4)–(5) how a large deviation problem ideally should be understood. To create a deviation $S_n \approx nu$ with $u > v$, the entire walk behaves as a random walk with mean step u . Namely, it can be proved that the conditioned measure $\mathbf{P}(\cdot | S_n \geq nu)$ converges on the path space to the distribution $\mathbf{Q}^{(u)}$ of a mean u random walk. The value $I(u)$ of the rate function in (5) is the entropy of this measure $\mathbf{Q}^{(u)}$ relative to the original \mathbf{P} .

Results of the type presented in this section appear for TASEP in (Seppäläinen, 1998a). The asymptotic analysis of Baik et al. (1999) and Johansson (2000) gives also LDP's for the growth models. Concentration results for a Brownian last-passage model appear in (Hambly et al., 2002) and a general discussion of deviation inequalities for growth models in the lectures of Ledoux (2007).

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